

Spherical Casimir energies and Dedekind sums

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Casimir energies on space-times having general lens spaces as their spatial sections are shown to be given in terms of generalised Dedekind sums related to Zagier's. These are evaluated explicitly in certain cases as functions of the order of the lens space. An easily implemented recursion approach is used.

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1. Introduction.

Exact solutions have a fascination that transcends any physical realisation. This is partly the reason for the use of toy, or cut down, models, the hope being that the generality and elegance of the solution would make up for any loss of realism. It is in this spirit that I have often resorted to evaluations in the Einstein Universe, which, although being static, is not a completely unrealistic cosmology, and it does form the instantaneous geometry of a Friedman–Robertson–Walker space–time.

In some works, [1,2], I considered the Casimir effect on the Einstein Universe, or three–sphere, looking at the factored case, S^3/Γ , and, in particular, at fixed–point free, homogeneous actions. Because of the geometry of rotations, it is sufficient to evaluate for lens spaces, *i.e.* for Γ a cyclic group, \mathbb{Z}_q , and combine these to get the other cases. This has been done in [2] where attention was restricted to one–sided (homogeneous) lens spaces, the simplest kind. In the present work, after a question by Gregory Moore and David Kutasov, I wish to extend my treatment to two–sided lens spaces the reason being that these are also exactly computable, in one sense. The answers are no doubt academic and are presented in accordance with the remarks above. However, there is current interest in the possibility of a topologically non–trivial Universe and my results may be of relevance here. Relatedly, there has been recent discussion of the implications of the S^3/Γ quotients for supersymmetry in the context of the AdS/CFT correspondance and the brane world scenario [3].

Because the physical motivation is meant only to lead quickly to the mathematical calculation, I concentrate on the total vacuum energy, as opposed to the energy density. In the next section some basic facts are outlined that lead as quickly as possible to the technicalities, which, I admit, are my main interest in this work, which should be considered as a continuation of [2].

One reason for the attraction of the three–sphere is that it is a group manifold. Because of the isomorphism $S^3 \sim \text{SU}(2)$, it is handy to implement the action of Γ on the sphere as the combination of left and right group actions on $\text{SU}(2)$ *i.e.* roughly, $\Gamma \sim \Gamma_L \times \Gamma_R$. A general point on the sphere S^3 is denoted by q , or, as an element of $\text{SU}(2)$, again by q (or sometimes by g) and the general action $q \rightarrow \gamma q$ is realised as $q \rightarrow \gamma_L q \gamma_R$.

2. Modes. Casimir energy.

I intend to consider (massless) fields of spin 0, 1/2 and 1. The spin zero expression for the Casimir energy has been given in [1], and repeated in [2], for double-sided actions. For one-sided actions, the corresponding formulae for spins 1/2 and 1 have been detailed in [4,5] and [6]. For completeness I wish to derive the double-sided forms. The answers could be deduced from the off-diagonal ζ -function given in [7] but it is preferable to start again. Unfortunately, a certain amount of dynamic and, worse, kinematic recapitulation is required to reach the needed forms.

The equations of motion that concern me here are the Klein-Gordon, Dirac and Maxwell equations. For mathematical convenience, I consider conformally coupled scalars so that the eigenvalues of the relevant Helmholtz equation are squares of integers.

In these cases there are no field theory divergences to worry about, in the absence of fixed points, and a reasonably unambiguous formula for the Casimir energy can be given as a regularised sum of the eigenvalues of the relevant Hamiltonian operator which, for scalars, is the square-root of the Helmholtz operator. The spectrum has positive and negative parts. In order to be more specific, some mode properties are required and I now briefly repeat some standard kinematics, [8,7], firstly on the full sphere.

For spinor fields (*i.e.* any field with spin greater than zero), the Cartan moving frame method projects everything onto one of the sets of Killing fields on S^3 . It is convenient to use the left-invariant set and we then say we are using ‘right spinors’. These are scalars under left actions and, further, the standard results of angular momentum theory can be used with no changes in standard conventions.

For spin-zero, the angular momentum representation matrices, $\mathcal{D}_N^{L M}(g)$, $L = 0, 1/2, 1, \dots$, provide a complete set of modes on the full three-sphere.

For higher spins, $j (= 1/2, 1, \dots)$, spin-orbit coupling gives the modes as the *spinor hyperspherical harmonics*,

$$\begin{aligned} Y_{mNJ}^{j LM}(q) &= \langle m, q | JMLjN \rangle \\ &= \left(\frac{(2L+1)(2J+1)}{2\pi^2 a^3} \right)^{1/2} \mathcal{D}_N^{LN'}(q) \begin{pmatrix} j & L & M \\ m & N' & J \end{pmatrix}. \end{aligned} \quad (1)$$

The equations of motion imply the massless polarisation conditions,

$$\begin{aligned} J &= L \pm j \quad \text{for } L \geq j \\ J &= L + j \quad \text{for } L < j, \end{aligned}$$

and then it can be shown by simple spin-orbit diagonalisation that the eigenvalues of the Hamiltonian operators (the square-root of the Helmholtz operator, the Dirac operator and the Maxwell curl) are,

$$H | JMLjN \rangle = E_{L,J}^j | JMLjN \rangle, \quad (2)$$

with

$$E_{L,L\pm j}^j = \frac{1}{a}(j \pm \bar{L}), \quad \bar{L} = 2L + 1.$$

The energy spectrum on the full sphere is thus

$$\begin{aligned} E_{\bar{L}}^+ &= \frac{1}{a}(j + \bar{L}), \quad \bar{L} = 1, 2, \dots \\ E_{\bar{L}}^- &= \frac{1}{a}(j - \bar{L}), \quad \bar{L} = 2j + 1, 2j + 2, \dots \end{aligned} \quad (3)$$

I note that E^+ is positive and E^- is negative, and that the spectrum on the full sphere is symmetrical about zero.

As is well known, factoring the sphere to S^3/Γ modifies the spectrum by eliminating certain modes and the spectrum is a subset of the full sphere one. I denote the resulting degeneracies of the eigenvalues (3) by $d^+(\bar{L})$ and $d^-(\bar{L})$ and define the handy positive and negative Hamiltonian ζ -functions,

$$\begin{aligned} \zeta_j^+(s) &= a^s \sum_{\bar{L}=1}^{\infty} \frac{d^+(\bar{L})}{(\bar{L} + j)^s} \\ \zeta_j^-(s) &= a^s \sum_{\bar{L}=2j+1}^{\infty} \frac{d^-(\bar{L})}{(\bar{L} - j)^s} = a^s \sum_{\bar{L}=1}^{\infty} \frac{d^-(\bar{L} + 2j)}{(\bar{L} + j)^s}, \end{aligned} \quad (4)$$

in terms of which the regularised value of the Casimir energy is

$$E_j = (-1)^{2j} \frac{1}{4} h(j) (\zeta_j^+(-1) + \zeta_j^-(-1)), \quad (5)$$

where $h(j)$ is a spin degeneracy factor, $h(0) = 1$ and $h(j) = 2$, $j > 0$. Corners have been cut here in anticipation of the fact that there is no divergence in the continuation of the ζ -functions at -1 .

The problem now falls onto the evaluation of the degeneracies which can be deduced in the following rather oblique way. Rather than analyse the projected modes, it is clearer to consider the covariant heat-kernel for, say the relevant second order spin- j propagation equation of which (1) are eigenmodes and the eigenvalues

are the squares of $E_{L,J}$, (2). The associated degeneracies will be denoted by $d(L, J)$ and are related to those in (4) by,

$$d^+(\bar{L}) = d(L, L + j), \quad d^-(\bar{L}) = d(L, L - j). \quad (6)$$

The spinor heat-kernel on S^3/Γ , $\mathbf{K}_\Gamma(q, q')$, is given, as usual, in terms of that, $\mathbf{K}(q, q')$, on S^3 by a pre-image sum. However, because right spinors have a non-trivial behaviour under right actions, it is necessary to provide compensating right ‘gauge’ rotations at the pre-image points to bring the beine into uniform alignment so that the pre-image sum makes sense.

One is therefore interested in the traced heat-kernel on the factored space,

$$K_\Gamma = \sum_{\gamma_L, \gamma_R} \int_{S^3/\Gamma} dq \operatorname{Tr} [\mathbf{K}(q, \gamma_L q \gamma_R) \mathcal{D}^j(\gamma_R^{-1})], \quad (7)$$

where \mathbf{K} is constructed from the eigenmodes (1). In order to use the properties of these, the integral in (7) is changed to one over the full sphere by using the invariance of the volume, dq , under symmetry actions and also the equation, following from the definition of right spinors, [8],

$$\mathbf{K}(\xi q \eta, \xi q' \eta) = \mathcal{D}^j(\eta^{-1}) \mathbf{K}(q, q') \mathcal{D}^j(\eta). \quad (8)$$

Then

$$K_\Gamma = \frac{1}{|\Gamma|} \sum_{\gamma_L, \gamma_R} \int_{S^3} dq \operatorname{Tr} [\mathbf{K}(q, \gamma_L q \gamma_R) \mathcal{D}^j(\gamma_R^{-1})]. \quad (9)$$

The switching equation, (8), shows that it is sufficient to know $\mathbf{K}(g) \equiv \mathbf{K}(g, \text{id})$ because $\mathbf{K}(g', g) = \mathbf{K}(g^{-1}g', \text{id})$ (note the order).

The construction of the heat-kernel, \mathbf{K} , in terms of the modes, Y , gives us the expression for the degeneracies ($|\mathcal{M}| = 2\pi^2 a^3$),

$$d(L, J) = \frac{\bar{L} \bar{J}}{|\Gamma| |\mathcal{M}|} \sum_{\gamma_L, \gamma_R} \begin{pmatrix} j & L & M \\ m & N & J \end{pmatrix} \begin{pmatrix} m' & N' & J \\ j & L & M \end{pmatrix} \mathcal{D}_{m'}^{j m}(\gamma_R) \int_{S^3} dq \mathcal{D}_{N'}^{L N}(\gamma_R q^{-1} \gamma_L q).$$

Group products and the orthogonality of the \mathcal{D} ’s easily yield

$$\int_{S^3} dq \mathcal{D}_{N'}^{L N}(\gamma_R q^{-1} \gamma_L q) = \frac{|\mathcal{M}|}{\bar{L}} \chi^{(L)}(\gamma_L) \mathcal{D}_{N'}^{L N}(\gamma_R)$$

and then standard angular momentum manipulations in the end give simply,

$$d(L, J) = \frac{1}{|\Gamma|} \sum_{\gamma_L, \gamma_R} \chi^{(L)}(\gamma_L) \chi^{(J)}(\gamma_R) \quad (10)$$

in terms of the SU(2) characters,

$$\chi^{(L)}(\gamma) = \frac{\sin \bar{L}\theta_\gamma}{\sin \theta_\gamma},$$

where θ_γ is the ‘radial’ angular coordinate labelling the group element, γ .

For spins greater than zero the spectrum is no longer symmetrical about zero, as is well known.

3. The Casimir energy formulae.

The equations (5), (4), (6) and (10) can now be put together to give explicit summations for the Casimir energy. Thus

$$E_j = \frac{(-1)^{2j}}{4|\Gamma|} h(j) \lim_{s \rightarrow -1} a^s \sum_{\gamma_L, \gamma_R} \left[\sum_{n=1}^{\infty} \frac{1}{(n+j)^s} \frac{\sin n\theta_L \sin(n+2j)\theta_R}{\sin \theta_L \sin \theta_R} + \sum_{n=2j+1}^{\infty} \frac{1}{(n-j)^s} \frac{\sin n\theta_L \sin(n-2j)\theta_R}{\sin \theta_L \sin \theta_R} \right],$$

where I have set $n = \bar{L}$ and $\theta_L = \theta_{\gamma_L}$, $\theta_R = \theta_{\gamma_R}$.

It must be admitted that this equation is meant to apply only for $j = 0, 1/2$ and 1. The further evaluation is possible for all j but the field theoretic significance is unclear.

The only n -sum in the group sum to diverge as $s \rightarrow -1$ is that for $\gamma = \text{id}$, ($\theta_R = \theta_L = 0$). So this term is separated and the limit taken in the rest.

The identity contribution is

$$E_j^{\text{id}} = \frac{(-1)^{2j}}{4|\Gamma|} h(j) \lim_{s \rightarrow -1} a^s \left[\sum_{n=1}^{\infty} \frac{n(n+2j)}{(n+j)^s} + \sum_{n=2j+1}^{\infty} \frac{n(n-2j)}{(n-j)^s} \right]$$

which can be evaluated as Hurwitz ζ -functions giving, expanding the Bernoulli polynomials, (see [7]),

$$E_j^{\text{id}} = \frac{(-1)^{2j}}{240a|\Gamma|} h(j) (30j^4 - 20j^2 + 1). \quad (11)$$

Specific values are

$$E_0^{\text{id}} = \frac{1}{240a|\Gamma|}, \quad E_{1/2}^{\text{id}} = \frac{17}{960a|\Gamma|}, \quad E_1^{\text{id}} = \frac{11}{120a|\Gamma|}, \quad (12)$$

which, apart from the volume factor, $1/|\Gamma|$, are the known full-sphere values.

The harder part of the calculation is the evaluation of the non-identity contribution,

$$\sum'_{\gamma_L, \gamma_R} \left[\sum_{n=1}^{\infty} (n+j) \frac{\sin n\theta_L \sin(n+2j)\theta_R}{\sin \theta_L \sin \theta_R} + \sum_{n=2j+1}^{\infty} (n-j) \frac{\sin n\theta_L \sin(n-2j)\theta_R}{\sin \theta_L \sin \theta_R} \right]. \quad (13)$$

It turns out that the integer and half-odd-integer j cases differ and so I treat them separately. For integer j , a shift of summation variable yields the more symmetrical form,

$$\sum'_{\gamma_L, \gamma_R} \frac{1}{\sin \theta_L \sin \theta_R} \sum_{n=j}^{\infty} n \left[\sin(n-j)\theta_L \sin(n+j)\theta_R + \sin(n+j)\theta_L \sin(n-j)\theta_R \right].$$

and simple trigonometry produces,

$$\sum'_{\gamma_L, \gamma_R} \frac{1}{\sin \theta_L \sin \theta_R} \sum_{n=j}^{\infty} n (\cos n\beta \cos j\alpha - \cos n\alpha \cos j\beta).$$

where

$$\alpha = \theta_R + \theta_L, \quad \beta = \theta_R - \theta_L.$$

The sum over n can be done,

$$-\frac{1}{2} \sum'_{\alpha, \beta} \left[\frac{\text{cosec}^2 \beta / 2 \cos j\alpha - \text{cosec}^2 \alpha / 2 \cos j\beta}{\cos \alpha - \cos \beta} \right].$$

While this expression can be taken further for any j , I now set just $j = 0$ and $j = 1$ for simplicity, when again simple algebra gives

$$-\frac{1}{2} \sum'_{\alpha, \beta} \left[\frac{\text{cosec}^2 \beta / 2 - \text{cosec}^2 \alpha / 2}{\cos \alpha - \cos \beta} \right] = -\frac{1}{4} \sum'_{\alpha, \beta} \text{cosec}^2 \beta / 2 \text{ cosec}^2 \alpha / 2, \quad (14)$$

for $j = 0$ and

$$\begin{aligned} & -\frac{1}{2} \sum'_{\alpha, \beta} \left[\frac{\text{cosec}^2 \beta / 2 \cos \alpha - \text{cosec}^2 \alpha / 2 \cos \beta}{\cos \alpha - \cos \beta} \right] = \\ & -\frac{1}{2} \sum'_{\alpha, \beta} \left(\text{cosec}^2 \alpha / 2 + \text{cosec}^2 \beta / 2 - \frac{1}{2} \text{cosec}^2 \alpha / 2 \text{ cosec}^2 \beta / 2 \right), \end{aligned} \quad (15)$$

for $j = 1$.

For spin-half I return to (13) and rewrite the sum as one over odd integers, $\bar{n} = 2n + 1$. Trigonometry and use of the α and β now give

$$\sum'_{\alpha,\beta} \frac{1}{\cos \alpha - \cos \beta} \sum_{\bar{n}=1,3}^{\infty} \bar{n} (\cos \bar{n}\beta/2 \cos \alpha/2 - \cos \bar{n}\alpha/2 \cos \beta/2),$$

and the \bar{n} summation is done using

$$\sum_{\bar{n}=1,3}^{\infty} \sin \bar{n}\theta = \frac{1}{2} \operatorname{cosec} \theta, \quad \sum_{\bar{n}=1,3}^{\infty} \bar{n} \cos \bar{n}\theta = -\frac{1}{2} \cot \theta \operatorname{cosec} \theta,$$

which yields,

$$\frac{1}{8} \sum'_{\alpha,\beta} \cot \alpha/2 \operatorname{cosec} \alpha/2 \cot \beta/2 \operatorname{cosec} \beta/2. \quad (16)$$

4. The Casimir energy calculated on lens spaces.

Reinstating the factors, the expressions for the three Casimir energies are

$$\begin{aligned} E_0 &= \frac{1}{a|\Gamma|} \left[\frac{1}{240} - \frac{1}{16} \sum'_{\alpha,\beta} \operatorname{cosec}^2 \beta/2 \operatorname{cosec}^2 \alpha/2 \right] \\ E_{1/2} &= \frac{1}{a|\Gamma|} \left[\frac{17}{960} + \frac{1}{8} \sum'_{\alpha,\beta} \cot \alpha/2 \operatorname{cosec} \alpha/2 \cot \beta/2 \operatorname{cosec} \beta/2 \right] \\ E_1 &= \frac{1}{a|\Gamma|} \left[\frac{11}{120} + \frac{1}{4} \sum'_{\alpha,\beta} (\operatorname{cosec}^2 \alpha/2 + \operatorname{cosec}^2 \beta/2 - \frac{1}{2} \operatorname{cosec}^2 \alpha/2 \operatorname{cosec}^2 \beta/2) \right]. \end{aligned} \quad (17)$$

The expression for E_0 was given in [1]. For one-sided actions (*i.e.* $\alpha = \pm\beta$), the expressions reduce to the ones in [2,6].

For the lens space, $L(q; l_1, l_2)$, the angles α and β take the values

$$\alpha = \frac{2\pi p\nu_1}{q}, \quad \beta = \frac{2\pi p\nu_2}{q}, \quad (18)$$

where $p, = 1, \dots, q-1$, labels γ . ν_1 and ν_2 are integers coprime to q , with l_1 and l_2 their mod q inverses. The simple, ‘one-sided’ lens space, $L(q; 1, 1)$, corresponds to setting $\nu_2 = \nu_1 = \nu = 1$, say, so that $\theta_L = 0$ and $\theta_R = 2\pi p/q$. These were discussed

in [2]. I now examine the general case and begin with the integer spin forms which involve only powers of cosecants. The single sums of cosec^2 are classic,

$$S(q; \nu) = \frac{1}{q} \sum_{p=1}^{q-1} \text{cosec}^2 \frac{\pi p \nu}{q} = \frac{q^2 - 1}{3q}, \quad (19)$$

and independent of ν , being sums over the roots of unity ordered in different ways as in the stellated polygon.

I require the harder, split summations,

$$S(q; \nu_1, \nu_2) = \frac{1}{q} \sum_{p=1}^{q-1} \text{cosec}^2 \frac{\pi p \nu_1}{q} \text{cosec}^2 \frac{\pi p \nu_2}{q} \quad (20)$$

where ν_1 and ν_2 are fixed integers from 1 to $q - 1$. For these particular forms I can fortunately make use of the computations of Zagier [9], who has defined a generalised Dedekind sum (I change his notation slightly)

$$\tau(q; \nu_1, \nu_2, \dots, \nu_n) = (-1)^{n/2} \frac{1}{q} \sum_{p=1}^{q-1} \cot \frac{\pi p \nu_1}{q} \dots \cot \frac{\pi p \nu_n}{q}. \quad (21)$$

Here q is a positive integer, ν_1, \dots, ν_n are integers prime to q and n is even. I note the important fact that one of the sets of roots of unity can be reordered to the canonical one while maintaining the relation between all the sets so keeping the sum the same. Formally this is done by multiplying all the ν_i by the mod q inverse, l_1 , say.

I need the four-dimensional case, $\tau(q; \nu_1, \nu_1, \nu_2, \nu_2)$, and, as said, one can put $\nu_1 = 1$ without loss of generality. So I look at $\tau(q; 1, 1, \nu, \nu)$ and note the simple relation that takes me back to the cosecant sums,

$$\begin{aligned} S(q; \nu, 1) &= \tau(q; 1, 1, \nu, \nu) + 2S(q; 1) - \frac{q-1}{q} \\ &= \tau(q; 1, 1, \nu, \nu) + \frac{(q-1)(2q-1)}{3q}, \end{aligned} \quad (22)$$

$S(q; 1)$ being given by (19).

In his Table 3, Zagier has listed some four-dimensional examples, of which $d(q; 1, 1, 3, 3)$ and $d(q; 1, 1, 4, 4)$ are most relevant for us. The first quantity has also been evaluated by Harvey *et al*, [10], in its cosec version. Employing Zagier's expression, one finds,

$$S(q; 3, 1) = \frac{q^4 + 210q^2 \pm 80q - 291}{405q}, \quad q = \pm 1 \pmod{3}$$

in agreement with [10]. Note that use of the cosec form has resulted in a more symmetrical result.

My intention here is to take the computations further for $\tau(q; 1, 1, \nu, \nu)$ with larger ν .

5. Evaluation of the Dedekind sums.

The essential calculational tool is the $(n + 1)$ -term reciprocity law obeyed by the Dedekind sums (21) (I state it for the 4-dimensional case),

$$\sum_{j=0}^4 \tau(\nu_j; \nu_0, \dots, \widehat{\nu_j}, \dots, \nu_4) = \phi_4(\nu_0, \dots, \nu_4), \quad (23)$$

where ν_0, \dots, ν_4 are pairwise coprime positive integers, the hat signifies omission of the indicated term and ϕ_4 is a function whose form can be calculated.

Zagier derives the relation (23) using contour integration which indicates why the construction of τ in terms of products of cotangents is so convenient as the pole defining function is another cotangent, giving a symmetrical structure. This leads to the expression for ϕ_n ,

$$\phi_n(\nu_0, \dots, \nu_n) = 1 - \frac{\mathcal{L}_n(\nu_0, \dots, \nu_n)}{\nu_0 \dots \nu_n}$$

where the polynomial, $\mathcal{L}_n(\nu_0, \dots, \nu_n)$ is the coefficient of t^n in the power series expansion of

$$\prod_{j=0}^n \frac{\nu_j t}{\tanh \nu_j t}.$$

The individual expansions of the coths give an explicit multiple sum expression for this quantity (Berndt, [11]) which is better expressed as a generalised Bernoulli polynomial but I do not need this cosmetically compact form here.

The relation (23) cannot be applied immediately to the case here because the members of the set, $(q, 1, 1, \nu, \nu)$, are not mutually coprime. However ν is prime to q so, following Zagier, I make use of periodicity to replace $\tau(q; 1, 1, \nu, \nu)$ by $-\tau(q; 1, 1, \nu, q - \nu)$ so that the set $(q, 1, 1, \nu, q - \nu)$ is coprime and (23) can be applied. In writing (23) out I immediately use the fact that $\tau(1; \nu_1, \nu_2, \nu_3, \nu_4)$ is always zero which produces a recursion formula, as is now shown. The reciprocal relation reduces directly to

$$\tau(q; 1, 1, \nu, q - \nu) + \tau(\nu; 1, 1, q, q - \nu) + \tau(q - \nu; 1, 1, \nu, q) = \phi_4(q, 1, 1, \nu, q - \nu) \quad (24)$$

For short, set

$$\tau_q = \tau(q; 1, 1, \nu, \nu)$$

so the first term on the left in (24) is $-\tau_q$ and the third term is $\tau_{q-\nu}$ as can be shown using mod $q - \nu$ periodicity. The second term is to be taken to the right-hand side and together with ϕ_4 will form the driving term of the recursion which, as is seen, successively reduces q by ν . The driving term has to be explicit. ϕ_4 is given by Zagier and equals, in this case,

$$\phi_4(q, 1, 1, \nu, q - \nu) = 1 - \frac{3\nu^4 - 6\nu^3q + 9\nu^2q^2 + 20\nu^2 - \nu q^3 - 20\nu q + 3q^4 + 20q^2 + 3}{45\nu q(q - \nu)} \quad (25)$$

A little more argument is needed for $\tau(\nu; 1, 1, q, q - \nu)$. It is necessary to break up the solution according to the residue classes mod ν of q . Thus, if $q = n\nu + x$, $\tau(\nu; 1, 1, q, q - \nu) = \tau(\nu; 1, 1, x, x)$ and this ‘constant’ has to be calculated separately for each x , ($1 \leq x \leq \nu$), for the given ν .

As an introduction to the general method, I check the case mentioned earlier, $\nu = 3$. There are two ranges for q , *i.e.* $q = \pm 1 \pmod{3}$. One therefore requires $\tau(3; 1, 1, 1, 1)$ and $\tau(3; 1, 1, -1, -1)$, $= \tau(3; 1, 1, 1, 1)$ also. This quantity follows from the classic summation,

$$\tau(q; 1, 1, 1, 1) = \frac{1}{q} \sum_{p=1}^q \cot^4 \frac{\pi p}{q} = \frac{(q-1)(q-2)(q^2 + 3q - 13)}{45q}, \quad (26)$$

as 2/27. Hence, evaluating the function ϕ_4 , (25), the recursion reads,

$$\tau_q + \frac{142}{135q} = \tau_{q-3} + \frac{142}{135(q-3)} + \frac{q^2}{45} - \frac{q}{15} - \frac{17}{45},$$

in both cases. This can be iterated down to $\tau_1 = 0$ for $q = 1 \pmod{3}$ and to $\tau_2 = \tau(2; 1, 1, 3, 3) = \tau(2; 1, 1, 1, 1) = 0$, also, for $q = -1 \pmod{3}$. Thus for $q = 3[q/3] + 1$,

$$\begin{aligned} \tau_q &= -\frac{142}{135q} + \sum_{k=1}^{[q/3]} \left(\frac{(3k+1)^2}{45} - \frac{3k+1}{15} - \frac{17}{45} \right) + \frac{142}{135} \\ &= \frac{(q-1)(q^3 + q^2 - 59q + 426)}{405q}, \end{aligned}$$

and for $q = 3[q/3] + 2$,

$$\begin{aligned} \tau_q &= -\frac{142}{135q} + \sum_{k=1}^{[q/3]} \left(\frac{(3k+2)^2}{45} - \frac{3k+2}{15} - \frac{17}{45} \right) + \frac{142}{270} \\ &= \frac{(q-2)(q^3 + 2q^2 - 56q + 213)}{405q}, \end{aligned} \quad (27)$$

which are the forms listed in Zagier, [9].

6. Automatic evaluation.

The above ‘hand’ calculation is typical. I wish to see how far it can be automated for any, but fixed, ν .

The form of the function ϕ_4 allows one to extract the combination,

$$\sigma_q = \tau_q + \frac{3\nu^4 + 20\nu^2 + 3}{45\nu^2 q}, \quad (28)$$

and to write the general recursion as,

$$\sigma_q - \sigma_{q-\nu} = \tau(\nu; 1, 1, x, x) - 1 + \frac{3q^2 - 3\nu q + 6\nu^2 + 20}{45\nu}.$$

For $q = [q/\nu]\nu + x$, for given x , this is taken down to σ_x to give ²

$$\begin{aligned} \sigma_q &= \sigma_x + \sum_{k=1}^{[q/\nu]} \left[\tau(\nu; 1, 1, x, x) - 1 + \frac{3(\nu k + x)^2 - 3\nu(\nu k + x) + 6\nu^2 + 20}{45\nu} \right] \\ &= \sigma_x + (\tau(\nu; 1, 1, x, x) - 1) \frac{(q - x)}{\nu} + \frac{(q - x)(q^2 + qx + x^2 + 5\nu^2 + 20)}{45\nu^2}. \end{aligned} \quad (29)$$

Everything is therefore explicit except the starting value, σ_x , and the ‘constant’, $\tau(\nu; 1, 1, x, x)$ which have to be found separately. The tables in Zagier could be consulted after suitable transformations using the basic properties of the τ ’s to reduce the arguments to those appearing in the table but I here continue with the recursion approach for a self-contained treatment.

Some streamlining of notation is indicated. I set,

$$\tau_q \equiv \tau(q; 1, 1, \nu, \nu) = \tau(q, \nu),$$

and rewrite the iteration result (29) in terms of the τ ’s,

$$\tau(q, \nu) = \tau(x, \nu) + \frac{(q - x)}{\nu} \tau(\nu, x) + F(q, \nu, x), \quad (30)$$

where,

$$\begin{aligned} F(a, b, c) &= \frac{1}{45b^2ac} \left(a(3b^4 + 20b^2 + 3) - ac^4 - 5ac^2(b^2 - 9b + 4) - \right. \\ &\quad \left. c(3b^4 + 5b^2(4 - a^2) + 45ba^2 - a^4 - 20a^2 + 3) \right). \end{aligned} \quad (31)$$

² x does not alter during the iteration.

I recall that q is arbitrary, ν is a fixed number, prime to q and

$$x = q \bmod \nu \quad (32)$$

is to be considered a chosen number, also prime to q .

In the first term on the right-hand side of (30), ν can be reduced mod x , defining,

$$x' = \nu \bmod x, \quad (33)$$

($x' < x < \nu < q$). Hence follows the fundamental relation,

$$\tau(q, \nu) = \tau(x, x') + \frac{(q-x)}{\nu} \tau(\nu, x) + F(q, \nu, x). \quad (34)$$

I introduce the series of Euclidean relations,

$$x_i = x_{i-2} \bmod x_{i-1}, \quad x_{i-1} < x_i, \quad (35)$$

of which (32) and (33) are the first two examples, with $x_0 = q$, $x_1 = \nu$, $x_2 = x$ and $x_3 = x'$. The recursion (34) then becomes, with a more systematic labelling,

$$\tau(x_i, x_{i+1}) = \frac{(x_i - x_{i+2})}{x_{i+1}} \tau(x_{i+1}, x_{i+2}) + \tau(x_{i+2}, x_{i+3}) + F(x_i, x_{i+1}, x_{i+2}), \quad (36)$$

which begins with $i = 0$ giving (34). The task, to repeat, is to find the ‘constants’ $\tau(x_1, x_2)$ and $\tau(x_2, x_3)$ by iterating (36).

The algorithm (35) ceases when one of the x_i reaches 1. Assume that this happens when $i = t$ (t is usually small). The iteration, (36), will also come to an end when $i + 2 = t$ because τ is known whenever one of the arguments is 1,

$$\begin{aligned} \tau(b, 1) &= \frac{(b-1)(b-2)(b^2+3b-13)}{45b} \\ \tau(1, d) &= 0. \end{aligned} \quad (37)$$

This means that $\tau(x_{t-2}, x_{t-1})$ is known from the recursion which can be rolled up from the bottom by successive back substitution. I find, for example, the first non-trivial value,

$$\begin{aligned} \tau(a, b) &= \frac{a-1}{b} \frac{(b-1)(b-2)(b^2+3b-13)}{45b} + F(a, b, 1) \\ &= \frac{(a-1)(a^3+a^2+a(b^4-15b^2-5)+3b^4+20b^2+3)}{45ab^2}, \end{aligned} \quad (38)$$

where $(a, b, 1)$ are the last three numbers in the Euclidean algorithm, (x_{t-2}, x_{t-1}, x_t) . In particular, if $a = b + 1$,

$$\tau(b + 1, b) = \frac{b(b - 1)(b^2 + 5b - 9)}{45(b + 1)},$$

which also follows simply from the reduction,

$$\begin{aligned}\tau(b + 1; 1, 1, b, b) &= \tau(b + 1; -b, -b, b, b) \\ &= \tau(b + 1; -1, -1, 1, 1) = \tau(b + 1; 1, 1, 1, 1),\end{aligned}$$

and (26).

Likewise, but more generally, one finds (see also the Appendix),

$$\tau(nb + 1; 1, 1, n, n) = \frac{nb(nb^3 + 4b^2 + nb(n^2 - 15) + 4n^2 + 5)}{45(nb + 1)}. \quad (39)$$

Similar reductions have to be performed in order to reach one of the given forms in Zagier's table. My formula has the manipulations built in.

The systematic elimination is therefore performed by inverse recursion and a more compressed notation is useful. I define

$$\begin{aligned}a_k &= \frac{x_{t+1-k} - x_{t+3-k}}{x_{t+2-k}} \\ \tau_k &= \tau(x_{t+1-k}, x_{t+2-k}) \\ f_k &= F(x_{t+1-k}, x_{t+2-k}, x_{t+3-k}),\end{aligned}$$

so that the two-term recursion (36) becomes

$$\tau_k = a_k \tau_{k-1} + \tau_{k-2} + f_k, \quad k = 3 \dots t, \quad (40)$$

with the starting values,

$$\begin{aligned}\tau_1 &= 0 \\ \tau_2 &= \frac{(b - 1)(b - 2)(b^2 + 3b - 13)}{45b}, \quad b = x_{t-1}.\end{aligned}$$

To make everything plain, my two step procedure is to compute the quantity I want $\tau(q, \nu) = \tau(x_0, x_1)$ from (36) at $i = 0$ by finding the constants $\tau(x_1, x_2)$ and $\tau(x_2, x_3)$ from the recursion (40), which has to be taken to $k = t - 1$ and $k = t$. The reason is that this second stage is purely numerical. The variable q is arbitrary, ν is a constant and x is a number chosen in the range $1 \leq x < \nu$. (It may be possible to leave x variable, but I prefer to set it by hand.)

The first act is the standard computation of the Euclidean algorithm (35) for given $x_1 = \nu$ and $x_2 = x$. It is then straightforward to implement the recursion to produce the polynomials in q in one go. I coded the entire process quite compactly in DERIVE.

7. Some results.

The general expression for the cosecant sum, $q S(q; \nu, 1)$, is a polynomial which I write in the following way,

$$q S(q; \nu, 1) = A(q^4 + Bq^2 + Cq + D), \quad (41)$$

where, introducing $x = q \bmod \nu$, the coefficients A, B, C, D depend on ν and x . A, B, D are unchanged under $x \rightarrow \nu - x$, while C changes to sign. So x can be restricted to $1 \leq x \leq \nu/2$, and, of course, x and ν are coprime.

When $x = 1$ or 2 , there are relations among the coefficients because the *cotangent* polynomials vanish when $q = 1$ or 2 , respectively.

The factor of A has been extracted because it is known from the obvious $q \rightarrow \infty$ limit of (20),³

$$A = 2 \frac{\zeta_R(4)}{\pi^4 \nu^2} = \frac{1}{45 \nu^2}. \quad (42)$$

(The factor of 2 is a consequence of the periodicity of the summand.) Hence only B, C and D need be displayed which I do in the following table for a few low values of ν and x ,

ν	x	B	C	D	ν	x	B	C	D
3	1	210	80	-291	9	1	7770	12320	-20091
4	1	490	360	-851	9	2	3450	4960	-20091
5	1	994	1008	-2003	9	4	3450	-640	-20091
5	2	706	144	-2003	10	1	11494	19008	-30503
6	1	1830	2240	-4071	10	3	4006	3456	-30503
7	1	3130	4320	-7451	11	1	16450	28080	-44531
7	2	1690	1440	-7451	11	2	6370	12240	-44531
7	3	1690	0	-7451	11	3	4930	6480	-44531
8	1	5050	7560	-12611	11	4	4930	3600	-44531
8	3	2170	1080	-12611	11	5	6370	-2160	-44531

The lowest case, $\nu = 2$, for which $B = 70$, $C = 0$ and $D = 71$ can be obtained directly using the relation $\operatorname{cosec}^2 2\theta = (\operatorname{cosec}^2 \theta + \sec^2 \theta)/4$ and the classic, single summations of cosec^4 , cosec^2 and \tan^2 .

³ This can be applied also to the general sum, (21).

8. Some Casimir values.

In this paper I consider only the integer spins, 0 and 1. The basic expressions for the Casimir energies are given in (17).

In terms of the sums discussed above, the spin 0 formula on the lens space $L(q; 1, l)$, where l is the mod q inverse of ν , is,

$$E_0(q, \nu) = \frac{1}{a} \left[\frac{1}{240q} - \frac{1}{16} S(q; \nu, 1) \right], \quad (43)$$

and for spin-1,

$$E_1(q, \nu) = \frac{1}{2a} \left[\frac{11}{60q} + S(q; 1) - \frac{1}{4} S(q; \nu, 1) \right]. \quad (44)$$

As a sample, I plot in fig.1 the spin-0 energy as a function of the order, q , for a fixed twisting, $\nu = 5$,

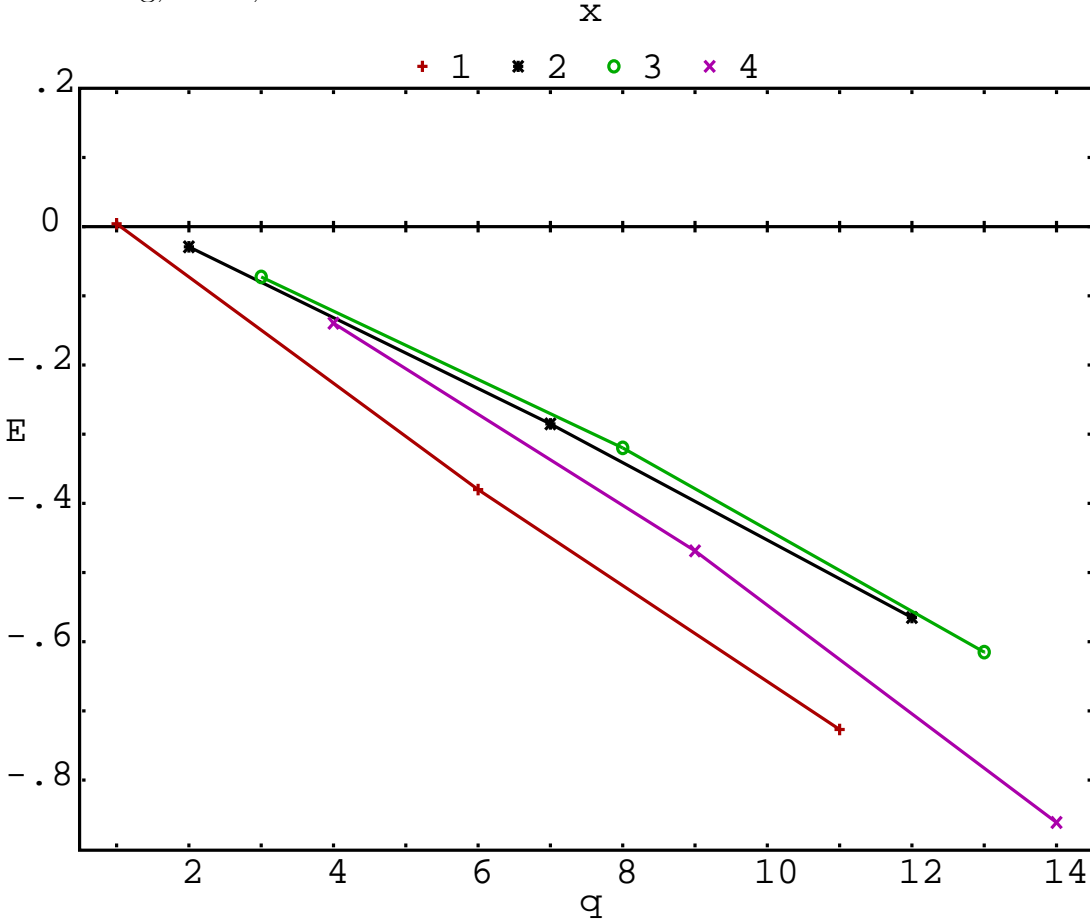


fig1. Casimir energy E for conformal scalars on lens spaces of order q and twisting $\nu = 5$. $x = q \bmod 5$

Fig.2 plots, typically, the Casimir energy against the twisting ν for a fixed order $q = 29$. It exhibits the symmetry about the value $\nu = q/2$ and a series of minima the significance of which I do not know. The spin-one graph is similar except it goes positive for certain values of ν .

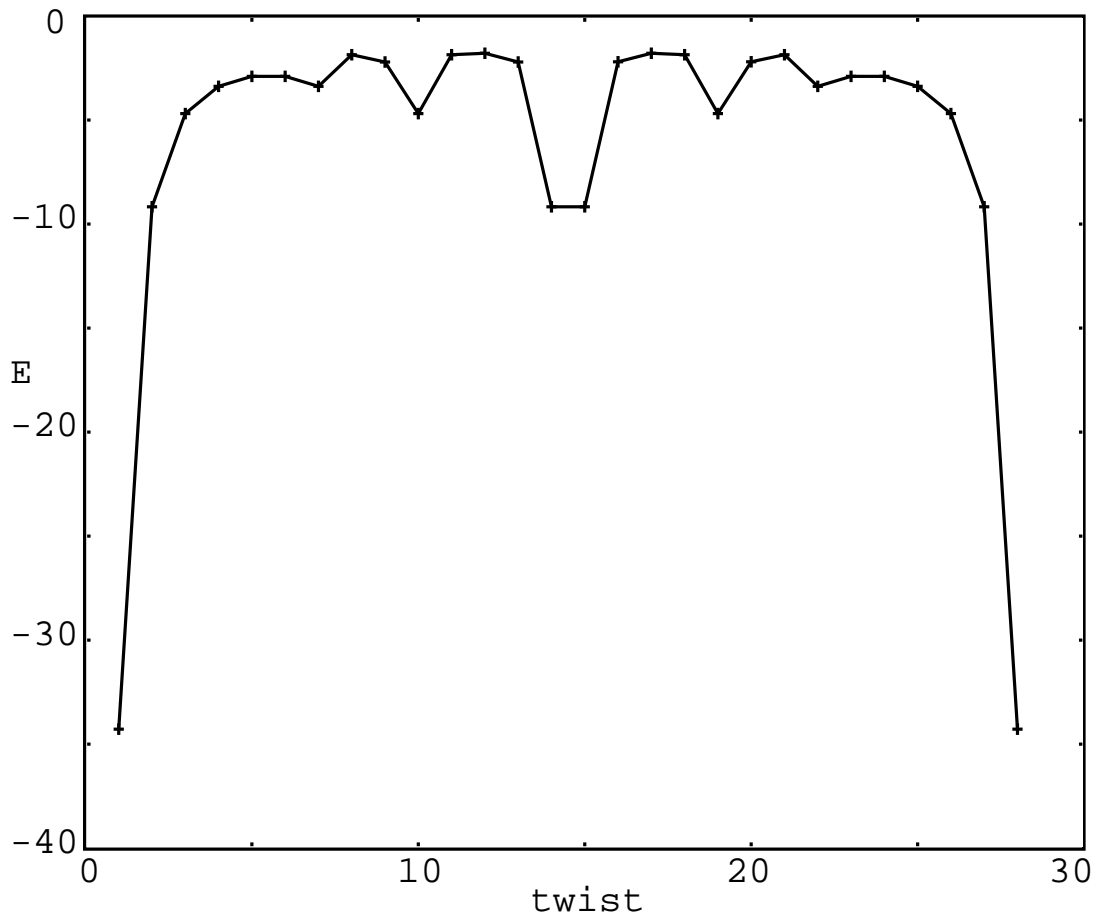


fig2. Casimir energy E for conformal scalars on a lens space of order 29 for twistings $\nu = 1$ to 28.

9. Conclusion and discussion.

The values for the Casimir energies presented here, and in our earlier works, are doubtless unpractical but their exact determination is not without methodological value.

Harvey *et al*, [10], present an algorithm for the computation of the cosecant sums, which can also be automated. In content, it is, of course, the same as that given here. However I feel the present approach via the cotangent sums is preferable as being more explicit and systematic.

There exist many generalisations of the original Dedekind sum. Some of the older ones are covered in the book by Rademacher and Grosswald, [12]. A more recent sum, extending that of Zagier, has been proposed by Beck, [13], which involves replacing each cotangent in (21) by a higher derivative. Beck employs contours to determine the reciprocity relation which could be used to compute the cosecant sums directly by simply restricting to first derivatives. I have not checked this. The article by Berndt and Yeap, [14], contains further references.

Unfortunately I could not find the spin-half summations, (16), so their reciprocal relation will have to be sought using contours, *ab initio*.

Also, as a challenge, if one could evaluate the even higher expressions,

$$\tau(q; 1, 1, \nu, \nu, \dots, \mu, \mu) = \frac{1}{q} \sum_{p=1}^{q-1} \cot^2 \frac{\pi p}{q} \cot^2 \frac{\pi p \nu}{q} \dots \cot^2 \frac{\pi p \mu}{q}, \quad (45)$$

the cosecant sums,

$$\frac{1}{q} \sum_{p=1}^{q-1} \operatorname{cosec}^2 \frac{\pi p}{q} \operatorname{cosec}^2 \frac{\pi p \nu'}{q} \dots \operatorname{cosec}^2 \frac{\pi p \mu'}{q}, \quad (\nu', \dots, \mu') \subset (\nu, \dots, \mu), \quad (46)$$

would follow on binomial expansion. Such sums would be needed when evaluating Casimir energies on the higher spheres.

For the case here, a more careful independent analysis of the asymptotic behaviour of the cosecant, or cotangent sums as $q \rightarrow \infty$ would yield general information about the coefficients in (41).

Appendix.

I make some remarks on the evaluation of the Dedekind sums not necessary for the immediate prosecution of the calculation of the Casimir energies but which extend the results a little.

In the preceding calculation the Euclidean algorithm was taken down to its $x_t = 1$ limit, which always works. However it is possible to stop if the penultimate value, x_{t-1} , equals 2 because τ is also known when one argument is 2,

$$\begin{aligned}\tau(x_{t-1}, x_t) &= \tau(2, 1) = 0 \\ \tau(a, 2) &= \frac{(a-1)(a^3 + a^2 - 49a + 131)}{180a}.\end{aligned}$$

The second expression checks against the result of the earlier recursion, and is in Zagier's list, but the point is that it follows *directly* from the standard sums of (even) powers of cotangents (or of cosecants) after using the simple trigonometric formula,

$$2 \cot \theta \cot 2\theta = \cot^2 \theta - 1. \quad (47)$$

For convenience, I denote the last four entries in the Euclidean algorithm, $(x_{t-3}, x_{t-2}, x_{t-1}, x_t)$, by (d, a, b, c) with $b = 2, c = 1$. The algorithm implies that a is odd, $a = 2n + 1$, and also that $d = ma + b$, ($n, m \in \mathbb{Z}$).

The first non-trivial result of the backwards recursion is

$$\tau(d, a) = \frac{d-b}{a} \tau(a, b) + F(d, a, b),$$

or, putting in the values,

$$\begin{aligned}\tau(m(2n+1) + 2, 2n+1) \\ = \frac{m}{45(m(2n+1) + 2)} \left(m^3(4n^2 + 4n + 1) + 8m^2(2n+1) \right. \\ \left. + m(4n^4 + 8n^3 - 24n^2 - 28n + 4) + 16n^3 + 24n^2 + 2n - 3 \right).\end{aligned}$$

Setting $n = 1, 2, \dots$ reproduces a subset of our previous expressions. For example $n = 1$ gives (27). The advantage of this form, and its companion, (39), is the explicit dependence on n .

Incidentally, the identity (47) can be used to obtain the specific value of the original Dedekind sum, $s(2, q) = (q-1)(q-5)/24q$. More generally it allows

one to obtain explicit forms for the sums, $\tau(q; 1, \dots, 1, 2, \dots, 2)$, as combinatorial combinations of generalised Bernoulli polynomials in q yielding, for example,

$$\begin{aligned}\tau(q; 1, 1, 1, 2, 2, 2) &= -\frac{1}{7560q}(q-1)(q-5)(2q^4 + 12q^3 - 43q^2 - 318q + 995) \\ \tau(q; 1, 1, 1, 1, 2, 2) &= -\frac{1}{1890q}(q-1)(q-2)(q-4)(2q^3 + 7q^2 - 7q - 139) \\ \tau(q; 1, 1, 1, 1, 1, 2, 2, 2, 2, 2) &= \\ &= -\frac{1}{2993760q}(q-1)(q-5)(2q^8 + 12q^7 - 103q^6 - 678q^5 \\ &\quad + 2013q^4 + 15468q^3 - 18017q^2 - 185442q + 344425) .\end{aligned}$$

Curiously, the symmetrical quantities seem to vanish when $q = 5$ on the spheres S^3 , S^{11} , S^{19} ... but not on S^7 , S^{15} ...

The identity also permits a sum such as $\tau(q; 1, 2, \nu, \dots, \mu)$ to be replaced by the ‘simpler’ ones, $\tau(q; 1, 1, \nu, \dots, \mu)$ and $\tau(q; \nu, \dots, \mu)$. More pertinent for this paper, it means that $\tau(q; 1, 2, \nu, \nu)$ can be reduced to $\tau(q; 1, 1, \nu, \nu)$ (q odd), computed previously, and to $\tau(q; \nu, \nu) = \tau(q; 1, 1)$, a standard sum,

$$\tau(q; 1, 2, \nu, \nu) = \frac{1}{2}\tau(q, \nu) - \frac{1}{6}(q-1)(q-2) .$$

The table in Zagier enables one to check this relation, which I would like to regard as extending the scope of my earlier results.

The consequences of more complicated trigonometric identities are discussed by Zagier, [9]. He shows, for example, that $\tau(q; 1, 1, 2, 4) = \tau(q; 1, 2, 2, 2)$ by an identity following from (47). Actually, repeated direct application of just (47) reduces each of these quantities ultimately to standard summations, so rendering Zagier’s earlier computation via reciprocity unnecessary.

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